On the universality of quantum logical neural networks

Adenilton J. da Silva and Teresa B. Ludermir
Centro de Informática
Universidade Federal de Pernambuco
Recife, Brazil
{ajs3, tbl}@cin.ufpe.br

Wilson R. de Oliveira
Departamento de Estatística e Informática
Universidade Federal Rural de Pernambuco
Recife, Brazil
wilson.rosa@gmail.com

Abstract—In this paper we investigate the computational power of the quantum weightless neural networks (q-WNN) and propose a novel quantum weightless neural node. The new quantum neuron is derived from the quantum probabilistic logic node (qPLN), which is a mathematical quantisation of the weightless neural node PLN. By a slight modification on the input lines of the qPLN node it is possible to define the quantum psi-neuron. We show in this paper that psi-neurons can be used to approximate any quantum circuit.

Keywords—Quantum Neural Networks; Weightless Neural Networks; Universality

I. INTRODUCTION

Quantum computation has been used to develop (quantum) algorithms more efficient than any known classical algorithm for certain problems. Some examples are the factoring algorithm [1] with polynomial time, the Grover’s search algorithm [2] with quadratic speedup over its classical counterpart. Another advantage of quantum computing is the increased storage capacity of associative memories. In a quantum associative memory one can store $2^n$ patterns with only $n$ quantum bits. One of the first quantum associative memories was proposed by Ventura in [3].

The concept of quantum neural networks was firstly proposed by Kak in [4]. Several quantum neural networks [5], [6], [7] and quantum learning algorithms [8], [7] have been proposed since Kak’s work. In this paper we analyse a quantum weightless neural network denoted quantum probabilistic logic node (qPLN) proposed in [9].

Weightless neural networks (WNN) were developed by Igor Aleksander [10] as an engineer tool for pattern recognition and they have several practical applications [11], [12]. The mathematical quantisation of the WNN was proposed in [9], where we further develop and investigate the q-WNN. An outstanding characteristic of WNN is their direct implementation in hardware and high speed of the learning process [13].

After showing that the networks of pPLN nodes proposed in [9] is not universal, we propose a new quantum weightless neuron denoted $|\psi\rangle$-neuron. The main characteristic of $|\psi\rangle$-neuron is its capacity to implement a universal set of quantum operators. The proposed $|\psi\rangle$-neuron is a generalisation of the qPLN neuron proposed in [9].

This paper is organised as follows: Section II presents the basic definitions for WNN, Section III presents the concepts of quantum computation, Section IV presents quantum weightless neural networks, Section V presents the results of this paper. We proof that qPLN cannot implement a set of universal quantum operators, we define a generalization of qPLN neuron denoted $|\psi\rangle$-neuron and we show that $|\psi\rangle$-neuron can implement a set of universal quantum operators. Finally, section VI is the conclusion.

II. WEIGHTLESS NEURAL NETWORKS

A weightless node has binary input and output and can compute any Boolean function of its input. The first weightless node proposed was the RAM node. An $n$ input RAM node has $2^n$ memory locations $C[a]$ addressed by an $n$-bit string $a = a_1 a_2 \cdots a_n$. A binary signal $x = x_1 x_2 \cdots x_n$ on the input lines will access only one of these locations resulting in the output $y = C[x]$ [13]. Each location $C[x]$ is capable of storing an $m$-bit number. Figure 1 displays a RAM node, where $d$ (used only in the learning phase) is the desired output and $s$ is a binary number determining if the neuron is in learning phase or utilization phase.

Learning in WNN is much simpler than the adjustment of weights. Learning in RAM node takes place simply by writing into the corresponding look-up table entries. Besides the simplicity of the RAM based nodes, the RAM based networks have good generalisation capabilities [11] and computational power [14].

\[
\begin{array}{c|c|c}
\hline
s & 11 \cdots 1 & C[2^n - 1] \\
\hline
 & 11 \cdots 0 & C[2^n - 2] \\
\hline
\vdots & \vdots & \vdots \\
\hline
00 \cdots 1 & C[1] \\
00 \cdots 0 & C[0] \\
\hline
\end{array}
\]

Figure 1. RAM Node

The Probabilistic Logic Node (PLN) consists of a RAM node, where now a 2-bit number is stored at the addressed
III. QUANTUM COMPUTATION

In this section we briefly introduce the concepts of quantum computing necessary for the remaining sections. For a more complete introduction on quantum computation and information see [16], or [17] for a Computer Science approach.

The quantum information unit is the quantum bit or "qubit". The state of a qubit is a superposition (linear combination) of the two computational-basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

(2)

where $\alpha$ and $\beta$ are complex coefficients (called probability amplitudes) constrained by the normalization condition: $|\alpha|^2 + |\beta|^2 = 1$; and $|0\rangle, |1\rangle$ are a pair of orthonormal basis vectors representing each classical bit, or "cbit", as column vectors.

Tensor products are used to represent more than one qubit: $|i\rangle \otimes |j\rangle = |i,j\rangle = |ij\rangle$, where $i, j \in \{0, 1\}$. Operations on qubits are carried out by unitary operators. So quantum algorithms on $n$ bits are represented by unitary operators $U$ over the $2^n$-dimensional complex Hilbert space which are presented as combinations (tensor product, composition, etc) of a few simple 1-qubit or 2-qubit operators called gate. This combinational representation of unitary operators is called quantum circuit.

In quantum computation, almost all operators on qubits are reversible, except those used in measurements. A measurement collapses the existing superposition and a measurement may cause the loss of the information carried by a qubit. A measurement using the computational basis collapses a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into $|0\rangle$ with probability $|\alpha|^2$ or into $|1\rangle$ with probability $|\beta|^2$.

Some common quantum gates and their corresponding matrix operators are:

$$
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I|0\rangle = |0\rangle \quad I|1\rangle = |1\rangle
$$

$$
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad X|0\rangle = |1\rangle \quad X|1\rangle = |0\rangle
$$

$$
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
$$

The identity operator $I$ does nothing; flip operator $X$ behaves as the classical NOT on the computational basis; Hadamard transformation $H$ generates superposition of states; Toffoli operator $T$ have 3 input qubits and 3 output qubits and flips the third qubit if $|ab\rangle = |11\rangle$ as show in Figure 2.

One of the main properties of quantum computation is the quantum parallelism. If one apply a quantum operator $U_f$ that implement a function $f(x)$ such that $U_f|x, 0\rangle = |x, f(x)\rangle$ in a state in superposition $|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i|x_i\rangle$, the value of $f(x)$ will be computed for all qubits $|x_i\rangle$. The resultant state will be $\sum_{i=0}^{n-1} \alpha_i|x_i, f(x_i)\rangle$.

Due to of the quantum parallelism, if one has a quantum neural network implemented as a quantum operator, then it will be possible to use states in the superposition to evaluate the output of all patterns in the training set, all at once in parallel. A drawback is that the individual results of this computation are not direct accessible, since measurement give just one result.

By unitarity the number of input lines equal the number of output ones. These lines are usually called registers and are numbered from the top to the bottom as first, second, etc. Another important difference between quantum and classic computation is the non-cloning property:

**Theorem 3.1:** An arbitrary unknown qubit cannot be copied [16].

IV. QUANTUM NEURAL NETWORKS

Quantum Neural Networks are expected to be more efficient than Classical Neural Networks. Several quantisations of weighted neural networks were proposed, including for weightless neural networks [9], [18], [19], [20]. Quantum weightless neural networks do not use non-linear activation function, like sigmoid or tangent hyperbolic. This is important because non-linear activation functions will hardly have an exact quantum analogous [5]. A learning algorithm for quantum weightless neural networks that uses quantum superposition was proposed in [20].

The values stored in a PLN node 0, 1 and $u$ are, respectively, represented by the qubits $|0\rangle$, $|1\rangle$ and $H|0\rangle = |u\rangle$. The probabilistic output generator of the PLN is represented as measurement outcome of the corresponding qubit. We obtain a node qPLN with the following output:

$$y = \begin{cases} 0, & \text{if } C|x\rangle = |0\rangle \\ 1, & \text{if } C|x\rangle = |1\rangle \\ random(0, 1), & \text{if } C|x\rangle = |u\rangle \end{cases}$$

A qPLN with $n$ inputs has $2^n$ matrices $A$ controlled by $n$ input qubits $|a_i\rangle = |a_1a_2\ldots a_n\rangle$. A quantum signal $|I\rangle = \sum_i \alpha_i|a_i\rangle$ in the input lines of the node it will activate the
matrices controlled by their $|a_i\rangle$, whose amplitude is not zero.

The matrix $A$ in Equation 4 defines a quantum operator over three qubits that simply flip the second qubit if the two first are in the state $|01\rangle$, it does nothing if the first qubits are in the state $|00\rangle$ and apply Hadamard operator if the two first operators are in the state $|10\rangle$. The $U$ operator in Equation 4 is an arbitrary quantum operator.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & H & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \quad \text{where} \quad A|000\rangle = |00\rangle I|0\rangle$$

$$A|010\rangle = |01\rangle X|0\rangle$$

$$A|100\rangle = |10\rangle H|0\rangle$$

$$A|100\rangle = |11\rangle U|0\rangle$$

**Definition 4.1:** A qPLN node with $n$ inputs is represented by the operator $N$ described in Equation (5). The inputs, selectors and outputs of $N$ are organized in three quantum registers $|i\rangle$ with $n$ qubits, $|s\rangle$ with $2^n$ qubits and $|o\rangle = |0\rangle$ with 1 qubit. The quantum state $|i\rangle$ describe qPLN input, and quantum state $|s\rangle|o\rangle$ describes qPLN state.

$$N = \sum_{i=0}^{2^n-1} |i\rangle_n |s_i\rangle_A |s_{i2}^{i1} \rangle_A$$

One example of qPLN network is presented in Figure 4, qPLN networks have pyramidal architecture and low connectivity, where $(i_1, i_2, i_3, i_4)$ is the input of the neuron. The configuration of a given qPLN network is represented by a string of qubits $(s_1, \ldots, s_m)$ representing network selectors, where $m$ is the number of neurons, $s_j$ represent the selectors of the neuron $N_j$.

The qPLN networks can be used to simulate classical PLN networks and quantum learning in qPLN networks can be done using the superposition based learning algorithm proposed in [20]. The quantum weightless neural networks maintain the properties of the classical WNN and they can perform quantum learning.

**V. Implementing a Set of Universal Quantum Operators**

In this section we show that qPLN networks cannot simulate any quantum circuit. In Definition 5.2 we define the $|\psi\rangle$-node that is a generalization of the qPLN node and we show in Theorem 5.1 that circuits composed by $|\psi\rangle$-nodes can simulate any quantum circuit. We start this section with the Definition 5.1 were we present the definition of a computationally universal set of quantum gates [21].

**Definition 5.1:** A set of quantum gates $C$ is said to be Computationally Universal if it can be used to approximate to within $\epsilon$ error any quantum circuit which uses $n$ qubits and $t$ gates with only polylogarithmic overhead in ($n$, $t$, $1/\epsilon$) [21].

The Toffoli and Hadamard operators form a universal set of operators for the quantum computation as showed by Aharonov [21] and Shi [22]. One can simulate the Toffoli and Hadamard operator with a neural network model to show that the neural network model is Computationally Universal. With this strategy we try to show circuits composed by qPLN nodes can simulate any quantum circuit.

Now observe that if a qPLN node with inputs $|\psi\rangle$ and $|\phi\rangle$ were to output $H|\phi\rangle$ in the third register, the non-cloning property, stated in Theorem 3.1, would be violated. The action of the neuron would take $|\psi\rangle|\phi\rangle|0\rangle$ into $|\psi\rangle|\phi\rangle H|\phi\rangle$ and by applying the operator $I \otimes I \otimes H$ would result in $|\psi\rangle|\phi\rangle|\phi\rangle$, i.e. a copy of the second input register would be generated in the third register for any quantum state. So from the non-cloning theorem we arrived at the following conclusion:

**Lemma 5.1:** qPLN nodes with the fixed input cannot implement Hadamard operator.

One can easily see that the Lemma 5.1 is also true for qPLN networks. Now we will define a new weightless neural networks model derived from the qPLN and with the capacity to simulate a set of computationally universal quantum gates.
To allow for the Hadamard operator to be implemented one must make the fixed input $|0\rangle$ in the third input register of the $A$ operator an arbitrary valued register. Note that by doing so, if a state not in the canonical basis is fed into the third register, the nodes do not behave exactly as their classical counterpart. The node in which an arbitrary valued third register is allowed is called as $|\psi\rangle$-node.

**Definition 5.2:** A $|\psi\rangle$-node with $n$ inputs is represented by the operator $N$ described in Equation (5). The inputs, selectors and outputs of $N$ are organized in three quantum registers $|i\rangle$ with $n$ qubits, $|s\rangle$ with $2^n$ qubits and $|o\rangle$ with 1 qubit. The quantum state $|i\rangle$ describe $|\psi\rangle$-node input, and quantum state $|s\rangle|o\rangle$ describes $|\psi\rangle$-node state. The output register can be initialized with an arbitrary qubit or used as one input register.

Now one can simulate Hadamard and Toffoli operators (a computationally universal set of quantum gates) with the $|\psi\rangle$-node to show that circuits composed by $|\psi\rangle$-nodes can approximate any quantum circuit.

**Theorem 5.1:** Circuits composed by $|\psi\rangle$-nodes can approximate any quantum function.

Proof: Without loss of generality, a node with 3 inputs is considered to simulate the Hadamard and Toffoli operators. For this, by the lemma 1, is necessary to consider that the qPLN doesn’t have fixed inputs. The measurements on the first and second registers are also relaxed, allowing the output being passed unchanged.

The exact configurations of the nodes for simulating the Hadamard and Toffoli operators are presented in the fig. 3 and 4, respectively. Note that in the figures by $X$, $I$ or $U$ is meant that the matrix is receiving selectors that makes it behaves as $X$, $I$ or $U$, respectively, where $U$ is an arbitrary unitary operator for the node.

![Figure 5](image)

**Figure 5.** $|\psi\rangle$-node simulating Hadamard operator

![Figure 6](image)

**Figure 6.** $|\psi\rangle$-nodes simulating Toffoli gate

The $|\psi\rangle$-nodes can simulate a set of computationally universal quantum gates $\{H, T\}$. Then, by Definition 5.1 circuits composed by $|\psi\rangle$-nodes can approximate any quantum circuit. ■

In the proof of Theorem 5.1 we show that circuits composed by $|\psi\rangle$-nodes can simulate any quantum circuit. One problem in Theorem 5.1 is that the neural network topology is not considered. So we cannot affirm that $|\psi\rangle$-networks with pyramidal topology are Computationally Universal. Another problem appears in Definition 5.2. The output register $|o\rangle$ needs initialization or can be used as an input register, it is necessary to determine a strategy to choose this new parameter.

**VI. Conclusion**

In this paper we showed that the quantum probabilistic logic nodes is not universal we defined the $|\psi\rangle$-node. A proof that the quantum $|\psi\rangle$-nodes can implement a universal set of quantum circuits is given, i.e., this node can be used to create circuits which approximate any quantum circuit with arbitrary precision. Combined with the learning capability of the qPLN a universal quantum learning system can be constructed with the model proposed.

A learning algorithm to train quantum weightless neural networks called superposition based quantum learning algorithm (SSLA) was proposed in [20]. One can verify if is possible to train the $|\psi\rangle$-neuron with SSLA and investigate the generalization capabilities of a neural network composed by $|\psi\rangle$-neurons. The main problem to adapt SSLA to train $|\psi\rangle$-networks is how to initialize the quantum register $|0\rangle$ of the $|\psi\rangle$-neurons.

Another future work is to develop quantum learning algorithms that uses $|\psi\rangle$-neurons in superposition. For example, one can do this task using a quantum associative memory [23] to present the patterns in superposition and to recover the parameters of the network.

**Acknowledgments**

The authors would like to thank CNPq and FACEPE (Brazilian research agencies) for their financial support.

**References**


