Abstract. We adopt the derivative for fuzzy functions obtained via Zadeh’s extension of the classical derivative operator [1] and implement it in dynamical systems. Particularly, we explicit solutions to fuzzy initial value problems (FIVPs) that preserve the main properties and characteristics of functions of the base space, as periodicity and stability. This is a known feature of fuzzy differential inclusions (FDIs). However, unlike solving inclusions, we study a theory for fuzzy differential equations (FDEs). Some examples are provided to illustrate the theory and the solutions to FIVP are compared with those from other approaches.

Keywords: Fuzzy Derivative, Fuzzy Differential Equation, Zadeh’s Extension, Fuzzy Arithmetic

1 Introduction

Zadeh’s extension is considered a very powerful technique in fuzzy sets theory. It is a case of united extension, that is, it extends functions whose arguments are points to functions whose arguments are sets. Denoting by $\mathcal{F}(E)$ the space of fuzzy subsets of a topological space $E$, Zadeh’s extension, in particular, returns functions over the space $\mathcal{F}(E)$, given that these functions were originally defined over the space $E$. What is interesting is that such extended functions inherit the main properties and characteristics of the original function.

In fuzzy dynamical systems, many approaches make use of Zadeh’s extension. One example is the approach first proposed by Oberguggenberger [2] and later studied by Mizukoshi et al. [3], consists of solving an initial value problem (IVP) and extending the solution according to fuzzy parameters. It has been proved [3] that under certain conditions, these solutions have the same attainable sets of those obtained via FDIs.

It is also a common practice to define the field of a FIVP as Zadeh’s extension of some classical function. Chalco-Cano et al. [4] claim that this is an optimal interpretation and argue for the resulting arithmetic.

The usual fuzzy interval arithmetic (Moore’s interval arithmetic) is very simple to compute, since it operates only with interval endpoints. However, it leads...
to overestimations in general. Zadeh’s extension, on the other hand, defines a fuzzy arithmetic equivalent to using constraint interval arithmetic and extending it to fuzzy intervals [4].

By studying FIVPs via the Generalized Hukuhara derivative (G-derivative) with the field given by Zadeh’s extension, [4] intended to avoid the undesired property resulting from blending Moore’s interval arithmetic with the Hukuhara derivative (H-derivative). In the latter, it is well known that the diameter of the solution of a fuzzy initial value problem is always non-decreasing [5].

In this article we study the FIVPs obtained by applying Zadeh’s extension to classical IVPs. Different from [4], not only the field is extended, but also the operator derivative is obtained via fuzzification of the classical derivative.

To start with, it is important to define the type of fuzzy function that we operate with. Let $F(E)$ stand for the space of fuzzy subsets in a given topological space $E$ with nonempty compact $\alpha$-levels. For our purposes, we define fuzzy function as the fuzzy subset of a classical function space. In other words, a fuzzy function is an element $X$ that belongs to $F(E)$, where $E$ is a function space. We are specially concerned when $E$ is the space of absolutely continuous functions, which we denote $AC([0,T];R^n)$. The H-derivative and the G-derivative, in turn, operate with functions of type $X : [0,T] \rightarrow F(R^n)$.

In order to compare these two kind of fuzzy functions, $X \in F(AC([0,T];R^n))$ and $X : [0,T] \rightarrow F(R^n)$, for each $X \in F(AC([0,T];R^n))$ we define the attainable sets at time $t$, $X(t)$, as the fuzzy subset of $R^n$ whose $\alpha$-levels are

$$[X(t)]^\alpha = [X]^\alpha(t) = \{x(t) : x \in [X]^\alpha \} \subset R^n.$$  

We investigate a concept of derivative for fuzzy function which is similar to that suggested by Chang and Zadeh [6] and later briefly studied by Dubois and Prade [7]. We employ the interpretation of Barros et al. [1]: the idea is to extend the classical derivative operator via Zadeh’s extension (see Section 2 for more details).

Briefly, given an IVP

$$\begin{align*}
\begin{cases}
x'(t) = f(t, x(t)) \\
x(0) = x_0
\end{cases},
\end{align*}$$

where $x_0 \in R^n$ and $f : R \times R^n \rightarrow R^n$, our purpose is to explore the FIVP

$$\begin{align*}
\begin{cases}
\hat{D}X(t) = \hat{f}(t, X(t)) \\
X(0) = X_0
\end{cases}
\end{align*}$$

where $X_0 \in F_K(R^n)$, $\hat{f} : R \times F_K(R^n) \rightarrow F_K(R^n)$ with $\hat{f}$ obtained by applying Zadeh’s extension to the second argument of a continuous function $f : R \times R^n \rightarrow R^n$ (or $f : R \times F_K(R^n) \rightarrow F_K(R^n)$) and $\hat{D}$ stands for the derivative (see Section 2).

Barros et al. [1] propose this kind of FDEs in a more general context and prove an existence theorem. As we shall see, this approach, in fact, leads to a theory of FDE, as does the H-derivative [8] and the G-derivative [9]. The solutions, however, can be related to those from FDIs [10], [11], [12].
2 Development

In abstract spaces \( U \) and \( V \), given a function \( f : U \rightarrow V \), we adopt the definition

\[
\tilde{f}(u)(y) = \begin{cases} 
sup_{s \in f^{-1}(y)} u(s), & \text{if } f^{-1}(y) \neq \emptyset \\
0, & \text{if } f^{-1}(y) = \emptyset 
\end{cases}
\]

for Zadeh’s extension. Thus, \( \tilde{f} \) is a fuzzy function such that \( \tilde{f} : F(U) \rightarrow F(V) \).

This means that \( f(x) \) belongs to \( \tilde{f}(u) \) with the same the membership as much as \( x \) belongs to \( u \), provided that \( f \) is an injective function.

It is well established that if \( f \) is a continuous function then \( [\tilde{f}(A)]^\alpha = f([A]^\alpha) \) for each \( \alpha \in [0, 1] \) (for more details, see [13]). According to Cecconello [14], this result is valid if the base space is a Hausdorff space.

When \( f : U \rightarrow F(V) \), Chang and Zadeh [6] and Huang and Wu [15] define \( \tilde{f} : F(U) \rightarrow F(V) \) such that

\[
\tilde{f}(u)(y) = \sup_{x \in U} \{ f(x)(y) \wedge u(x) \} .
\]

(4)

As we shall see shortly, this definition of extension will play an important role in this paper.

Provided the metric

\[
d_\infty(u, v) = \sup \{d_H([u]^\alpha, [v]^\alpha) : 0 \leq \alpha \leq 1 \},
\]

in \( F_K(R^n) \), [15] also states the following theorem.

**Theorem 1.** [15]

Let \( f : R \rightarrow F(R) \) be a \( d_\infty \)-continuous function. Then \( \tilde{f} \) is \( d_\infty \)-continuous and \( [\tilde{f}(u)]^\alpha = \bigcup_{x \in [u]^\alpha} [f(x)]^\alpha \).

**Derivative of fuzzy function**

We will use \( D \) to represent the operator derivative, i.e.,

\[
D : AC([0, T]; R^n) \rightarrow L^\infty([0, T]; R^n)
\]

\( w \rightarrow Dw = w' \)

where \( w' \) is the derivative in the sense of distributions (see [16]). Thus, there exists \( Dw(t) \) a.e., in \( [0, T] \).

**Definition 1.** Derivative of fuzzy function

Let \( W \in F(AC([0, T]; R^n)) \) be a fuzzy function. We define \( \tilde{D}W \) as the derivative of \( W \), where \( \tilde{D} \) is given by Zadeh’s extension of operator \( D \), according to formula (3).

Hence, \( w' \) belongs to \( \tilde{D}(W) \) with the same membership that \( w + k \) belongs to \( W \), for some \( k \in R^n \). Since the operator \( D \) is not continuous for uniform convergence (sup norm), the identity \( [\tilde{D}W]^\alpha = D([W]^\alpha) \) is not immediate. Still, since \( D \) is a closed operator, this property can be proved [1].
Theorem 2. [1]

Let \( W \in \mathcal{F}_K(\mathcal{AC}([0,T];\mathbb{R}^n)) \). Then

\[
[D(W)^\alpha] = D([W]^\alpha).
\]

Fuzzy differential equations

Consider the FIVP

\[
\hat{D}X(t) = F(t, X(t)) \\
X(0) = X_0
\]

(5)

where \( F : \mathbb{R} \times \mathcal{F}_K(\mathbb{R}^n) \to \mathcal{F}_K(\mathbb{R}^n) \) and \( X_0 \in \mathcal{F}_K(\mathbb{R}^n) \).

A solution to (5) is a fuzzy function \( X(\cdot) \in \mathcal{F}_K(\mathcal{AC}([0,T];\mathbb{R}^n)) \) that satisfies (5) a.e. in \([0,T]\).

If \( F \) meets certain conditions, including

\[
[F(t,X)]^\alpha = \bigcup_{x \in [X]^\alpha} [F(t,x)]^\alpha
\]

(6)

the existence of a solution to (5) is assured [1]. In fact, it was proved that the solution of the FDI associated is also a solution to (5). In particular, if \( F \) is Zadeh’s extension of some classical function, (6) is satisfied. Actually (see Theorem 1), if a fuzzy-valued function is \( d_\infty \)-continuous, then its extension in the sense of (4) satisfies (6).

In this paper we consider the field \( F \) as Zadeh’s extension, i.e., we deal with FIVPs of type

\[
\hat{D}X(t) = \hat{f}(t, X(t)) \\
X(0) = X_0
\]

(7)

where \( X_0 \in \mathcal{F}_K(\mathbb{R}^n) \) and \( \hat{f} : \mathbb{R} \times \mathcal{F}_K(\mathbb{R}^n) \to \mathcal{F}_K(\mathbb{R}^n) \) in which \( \hat{f} \) is Zadeh’s extension of \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) (or \( f : \mathbb{R} \times \mathbb{R}^n \to \mathcal{F}_K(\mathbb{R}^n) \) like in formula (4)).

From Theorem 2, in \( \alpha \)-levels, (7) is equivalent to the family of IVPs

\[
[D[X]^\alpha(t)] = [\hat{f}(t, X(t))]^\alpha \\
[X(0)]^\alpha = [X_0]^\alpha
\]

(8)

for all \( \alpha \in [0,1] \).

We present some examples in which the field is given by Zadeh’s extension of some function with real-valued argument. It is worth mentioning that, when we write equations such as \( X + X^2 \), we do not mean that we have applied Moore’s interval arithmetic to the \( \alpha \)-levels. This is simply a notation for extending the function \( x + x^2 \). Details of the resulting arithmetics can be found in [4] and [17].

Example 1. Kaleva [5] considers the FIVP

\[
\begin{align*}
X'(t) &= X(t)^2 \\
X(0) &= X_0
\end{align*}
\]

(9)
where $X_0$ is the triangular fuzzy number $(1, 2, 3)$.

- **FDI**

  We will first find a solution to the FDI associated to (9):

  $$\begin{cases}
  x'(t) = x(t)^2 \\
  x_0 \in [1 + \alpha, 3 - \alpha]
  \end{cases}$$

  (10)

  where $f$ is the continuous function $f(x) = x^2$, for each $\alpha \in [0, 1]$.

  The solution to (10) has the $\alpha$-levels

  $$[X(\cdot)]^\alpha = \left\{ x(\cdot) : x'(\cdot) = (x(\cdot))^2, x_0 \in [1 + \alpha, 3 - \alpha] \right\}$$

  (11)

  $$\begin{cases}
  x(\cdot) = \frac{x_0}{1 - x_0 t}, x_0 \in [1 + \alpha, 3 - \alpha] \\
  \end{cases}.$$

  (12)

- **H-derivative**

  According to [5], the solution via the H-derivative calculated in time $t$ is equal to the attainable set in $t$ of the solution to the FDI associated (10).

  The attainable fuzzy sets, $X(t)$, are given by the $\alpha$-levels

  $$[X(t)]^\alpha = \left\{ \frac{x_0}{1 - x_0 t}, x_0 \in [1 + \alpha, 3 - \alpha] \right\}$$

  $$= \left[ \frac{1 + \alpha}{1 - (1 + \alpha) t}, \frac{3 - \alpha}{1 - (3 - \alpha) t} \right],$$

  (13)

  because the function $\frac{x_0}{1 - x_0 t}$ is continuous with regard to $x_0$, if $0 \leq t < 1/3$.

- **$\hat{D}$-derivative**

  Since the multiplication in $X \times X$ is interpreted as extension of $f(x) = x^2$, it is possible to verify that (11) is also a solution to (9) with the $\hat{D}$-derivative. That is, the attainable sets $X(t)$ of both solutions, via FDI and $\hat{D}$-derivative, are the same. And these attainable sets have the same $\alpha$-levels of the solution via the H-derivative in time $t$. These solutions are illustrated in Figure 1.

*Example 2.* Consider a decay model, in which the parameter is uncertain and modeled in a fuzzy context:

$$\begin{cases}
  X'(t) = -AX(t) \\
  X(0) = x_0 \in R
  \end{cases}$$

(14)

with $A$ a fuzzy number whose $\alpha$-levels are $[A]^\alpha = [\lambda_1^\alpha, \lambda_2^\alpha]$ and $\lambda_1^\alpha > 0$, for all $\alpha \in [0, 1]$.

Note that $F(X) = -AX = \tilde{f}(X)$, where $\tilde{f}$ is extension of $f(x) = -Ax$ (according to [15]).

We will first find a solution to (14) via the H-derivative and then offer another by applying the $\hat{D}$-derivative.
Fig. 1. A solution to $\hat{D}X(t) = X(t)^2$ of Example 1 with the triangular fuzzy number $(1, 2, 3)$ as initial condition. The darker the color, the greater the membership of the point to the solution. The attainable sets of the solution obtained employing the $\hat{D}$-derivative are the same attainable sets of the solution via FDI. They also coincide with the solution using the H-derivative.

• H-derivative

For each $\alpha$-level, applying the H-derivative in (14) leads to the classical system

\[
\begin{cases}
(x_1^\alpha)'(t) = -\lambda_2^\alpha x_1^\alpha(t) \\
(x_2^\alpha)'(t) = -\lambda_1^\alpha x_2^\alpha(t) \\
 x_1^\alpha(0) = x_0 \\
 x_2^\alpha(0) = x_0
\end{cases}
\]  

whose solution is

\[
\begin{cases}
 x_1^\alpha(t) = c_1^\alpha e^{\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} t} + c_2^\alpha e^{-\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} t} \\
 x_2^\alpha(t) = -\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} c_1^\alpha e^{\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} t} + \sqrt{\lambda_1^\alpha / \lambda_2^\alpha} c_2^\alpha e^{-\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} t}
\end{cases}
\]  

with

\[ c_1^\alpha = \frac{\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} - 1}{2} x_0 \quad \text{and} \quad c_2^\alpha = \frac{\sqrt{\lambda_1^\alpha / \lambda_2^\alpha} + 1}{2} x_0. \]

Thus, the solution $X(t)$ of (14) via the H-derivative has the $\alpha$-levels $[X(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$.

• $\hat{D}$-derivative
To find the solution \( X(\cdot) \) via the \( \hat{D} \)-derivative, we calculated affine combinations of the endpoints \( \lambda_1^\alpha \) and \( \lambda_2^\alpha \):

\[
\beta_1^{(\alpha,\gamma)} = \gamma \lambda_1^\alpha + (1 - \gamma) \lambda_2^\alpha \quad \text{and} \quad \beta_2^{(\alpha,\gamma)} = \gamma \lambda_1^\alpha + (1 - \gamma) \lambda_2^\alpha
\]

with \( \gamma \in [0, 1] \) and \( \alpha \in [0, 1] \).

The idea here is to find curves in the format (16) such that they are between \( x_1^\alpha(t) \) and \( x_2^\alpha(t) \) of the solution via the H-derivative. It is also necessary that they fill the space between \( x_1^\alpha(t) \) and \( x_2^\alpha(t) \), that is, we want a compact subset of functions. For this, it is sufficient to use the affine combinations of \( \lambda_1^\alpha \) and \( \lambda_2^\alpha \) as coefficients of system (15). This technique is explored by [17] to represent fuzzy intervals.

Using this as new parameters for problem (15), we obtain the family of curves

\[
\begin{align*}
x_1^{(\alpha,\gamma)}(t) &= c_1^{(\alpha,\gamma)} \sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} - c_2^{(\alpha,\gamma)} e^{-\sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} t} + c_2^{(\alpha,\gamma)} e^{-\sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} t} \\
x_2^{(\alpha,\gamma)}(t) &= k^{(\alpha,\gamma)} \left( c_1^{(\alpha,\gamma)} e^{\sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} t} - c_2^{(\alpha,\gamma)} e^{\sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} t} \right)
\end{align*}
\] (17)

with

\[
k^{(\alpha,\gamma)} = \sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}}
\]

\[
c_1^{(\alpha,\gamma)} = \left( \sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} - 1 \right) x_0 \quad \text{and} \quad c_2^{(\alpha,\gamma)} = \left( \sqrt{\beta_1^{(\alpha,\gamma)} / \beta_2^{(\alpha,\gamma)}} + 1 \right) x_0.
\]

The union of all these curves produces the \( \alpha \)-levels of a fuzzy function \( X(\cdot) \) which is the solution of (14) employing the \( \hat{D} \)-derivative. In order to see this, note that, since each pair \( (x_1^{(\alpha,\gamma)}(t), x_2^{(\alpha,\gamma)}(t)) \) is solution to problem (15) with affine combinations of \( \lambda_1^\alpha \) and \( \lambda_2^\alpha \) as coefficients, the union of the derivative of all these curves is equal to \( \{-\beta_1^{(\alpha,\gamma)} x_1^{(\alpha,\gamma)} : 0 \leq \alpha, \gamma \leq 1; i = 1, 2\} \). Calculating the attainable sets in time \( t \) gives us \( \{-\beta_1^{(\alpha,\gamma)} x_1^{(\alpha,\gamma)}(t) : 0 \leq \alpha, \gamma \leq 1; i = 1, 2\} = [-A]^\alpha[-X(t)]^\alpha \), and so \( \hat{D}X(t))^\alpha = [-AX(t)]^\alpha \). Thus, in fact, \( X(\cdot) \) is a solution to

\[
\begin{align*}
\hat{D}X(t) &= -AX(t) \\
x(0) &= x_0
\end{align*}
\] (18)

The attainable sets in time \( t \) are the solution to (14) via the H-derivative calculated in time \( t \). This is due to the fact that the attainable sets of the \( \alpha \)-levels of the solution are the intervals \( [x_1^{(\alpha,0)}(t), x_2^{(\alpha,0)}(t)] \) given by (17), which are the same as \( [x_1^{(\alpha,0)}(t), x_2^{(\alpha,0)}(t)] \) given by (16).

Hence, in this case, the attainable sets of a solution to (18) are the same as those obtained via the H-derivative (illustrated in Figure 2).

At this point, it is worth mentioning that the solution found for (18) using the \( \hat{D} \)-derivative is not necessarily unique. This will be illustrated by solving another PVIF in the example below.
Example 3. Consider the decay model in which the initial condition is fuzzy:

\[
\begin{cases}
X'(t) = -\lambda X(t) \\
X(0) = X_0
\end{cases}
\tag{19}
\]

where \(X_0\) is a fuzzy number and \(\lambda > 0\).

- **FDI**
  
  Solving the associated FDI,

\[
\begin{cases}
x'(t) = -\lambda x(t) \\
x_0 \in [X_0]^{\alpha}
\end{cases}
\tag{20}
\]

for each \(\alpha \in [0, 1]\) we obtain \([X(\cdot)]^{\alpha} = \{x_0 e^{-\lambda t}, x_0 \in [x_0^{\alpha_1}, x_0^{\alpha_2}]\}\), whose attainable sets are \([X(t)]^{\alpha} = [x_0^{\alpha_1}, x_0^{\alpha_2}] e^{-\lambda t}\).

- **\(\hat{D}\)-derivative: first solution**
  
  From a direct calculation, one can confirm that \(X(\cdot)\) is also a solution to (19) employing the \(\hat{D}\)-derivative. Actually, the existence theorem in [1] guarantees this result.

- **H-derivative**
Using the H-derivative, we find a solution to (19) solving
\[
\begin{cases}
(x_1^\alpha)'(t) = -\lambda^\alpha x_2^\alpha(t) \\
(x_2^\alpha)'(t) = -\lambda^\alpha x_1^\alpha(t) \\
x_1^\alpha(0) = x_{01}^\alpha \\
x_2^\alpha(0) = x_{02}^\alpha
\end{cases}
\tag{21}
\]

So, the solution \(X(t)\) is such that \([X(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]\) with
\[
\begin{cases}
x_1^\alpha(t) = c_1^\alpha e^{\lambda t} + c_2^\alpha e^{-\lambda t} \\
x_2^\alpha(t) = -c_1^\alpha e^{\lambda t} + c_2^\alpha e^{-\lambda t}
\end{cases}
\tag{22}
\]
and
\[
c_1^\alpha = \frac{x_{01}^\alpha - x_{02}^\alpha}{2} \quad \text{and} \quad c_2^\alpha = \frac{x_{01}^\alpha + x_{02}^\alpha}{2}.
\]

- \(\hat{D}\)-derivative: second solution

There is another solution whose attainable sets coincide with the solution using the H-derivative.

To obtain a solution \(X(\cdot)\) to the approach using the \(\hat{D}\)-derivative, we construct \([X(t)]^\alpha\) by calculating affine combinations of the initial conditions in (21):
\[
z_{01}^\alpha = \gamma x_{01}^\alpha + (1 - \gamma)x_{02}^\alpha \quad \text{and} \quad z_{02}^\alpha = \gamma x_{01}^\alpha + (1 - \gamma)x_{02}^\alpha
\]
with \(\gamma \in [0, 1]\).

We employ the same procedure as in Example 2. We use the system obtained using the H-derivative to find curves to construct the solution for the \(\hat{D}\)-derivative.

Using \(z_{01}^\alpha\) and \(z_{02}^\alpha\) as new initial conditions for problem (21), we obtain a family of curves:
\[
\begin{cases}
x_1^{(\alpha, \gamma)}(t) = c_1^{(\alpha, \gamma)} e^{\lambda t} + c_2^{(\alpha, \gamma)} e^{-\lambda t} \\
x_2^{(\alpha, \gamma)}(t) = -c_1^{(\alpha, \gamma)} e^{\lambda t} + c_2^{(\alpha, \gamma)} e^{-\lambda t}
\end{cases}
\tag{23}
\]
with
\[
c_1^{(\alpha, \gamma)} = \frac{(1 - 2\gamma)x_{01}^\alpha - (1 - 2\gamma)x_{02}^\alpha}{2} \quad \text{and} \quad c_2^{(\alpha, \gamma)} = \frac{x_{01}^\alpha + x_{02}^\alpha}{2}
\]

This family compounds a solution that is also a solution to (19) with the \(\hat{D}\)-derivative. So, as mentioned previously, it has been shown that the \(\hat{D}\)-derivative does not lead to unique solutions to FDEs (see Figures 3 and 4).

This is not specific to the \(\hat{D}\)-derivative. The H-derivative is a particular case of the G-derivative. So, the solution obtained via the H-derivative is a solution to problem (19) using the G-derivative. It can be shown, however, that \([X(t)]^\alpha = [x_{01}^\alpha, x_{02}^\alpha] e^{-\lambda t}\) also satisfies (19) with the G-derivative [9].
Fig. 3. A solution to FIVP $\hat{D}X(t) = -\lambda X(t), X(0) = X_0$, from Example 3. $\lambda = \{2\}$ and $X_0$ is the triangular fuzzy number (0.5, 1, 1.5). The darker the color, the greater the membership of the point to the solution. The solution has the same attainable sets as the one obtained by FDIs.

Fig. 4. Another solution to the same FIVP $\hat{D}X(t) = -\lambda X(t), X(0) = X_0$, from Example 3. $\lambda = \{2\}$ and $X_0$ is the triangular fuzzy number (0.5, 1, 1.5). The darker the color, the greater the membership of the point to the solution. The solution has the same attainable sets as solution via H-derivative.
3 Conclusion

We explored FIVPs employing fuzzification of the classical derivative operator and called it the $\hat{D}$-derivative. Solutions to some FIVPs using the derivative have been proposed and compared with those from other approaches. It has also been shown that the property of uniqueness of the solution is not valid, even for a linear field (Example 3). Different solutions to the same problem were explicitly exhibited.

Unlike the approach based on differential inclusions (DIs), the $\hat{D}$-derivative enables us to develop a theory of FDEs. Such theory is not limited to this case. Barros et al. [1] has presented other cases and has also provided theoretical basis required in this study.

Our solutions $X$ belong to the space of fuzzy subsets of functions, that is, $X \in \mathcal{F}(E)$, where $E$ is a classical space of functions. Thus, we focussed in fuzzy functions that preserve the main properties and characteristics of functions of the base space, a feature of the theory of FDEs. An example of a desired property that does not occur with the H-derivative is a solution for a decay model going to zero when $t \to \infty$, for all $\alpha$-levels (see Figure 3). This leads us to conclude that it makes sense to analyse stability in FIVPs using the $\hat{D}$-derivative.

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